# Spatially Inhomogeneous Plasma with Negative Ions and Modified KdV Equation with $x$ Dependent Term

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We have derived a modified KdV equation with $x$ dependent term in the case of a spatially inhomogeneous plasma using the stretched co-ordinate system by Asano. Exact and perturbation solutions derived from inverse scattering method are discussed.

## Introduction

1. Extensive studies on ion-acoustic KdV and MKdV solitons on a homogeneous plasma have been made both theoretically and experimentally [1]. It is well known that an ion-acoustic wave is described by the KdV equation, which is given by

$$\frac{\partial \psi}{\partial \tau}+\psi \frac{\partial \psi}{\partial \xi}+\frac{1}{2} \frac{\partial^3 \psi}{\partial \xi^3}=0, \tag{i}$$

when $\psi$ is the wave potential and $\tau$ and $\xi$ are the time and pace coordinate, respectively.

However, for the propagation of ion-acoustic waves in a spatially inhomogeneous plasma (i) gets modified. Explicit solutions in this case have been obtained, and it was found that the width and velocity of the soliton differ from those obtained from (i) (see [2], also [3] and [4]). Introduction of negative ions modifies considerably the response of the plasma. In this paper we consider the propagation of an ion-acoustic wave in an inhomogeneous plasma composed of electrons and one species of positive and negative ions using the stretched co-ordinates as suggested by Asano [5], which are appropriate to derive the MKdV equation at the critical density of the negative ions.

This problem has been studied by Watanabe [6] for a homogeneous plasma. But our coordinates will be completely different from those of Watanabe, and hence the expansions of the physical quantities will also be different.

2. *The modified KdV equation for ion-acoustic waves in the presence of inhomogeneity.* In the homogeneous plasma with electrons and one species of positive and negative ions the ion-acoustic wave of weak nonlinearity is described by the following KdV equation (see [7]):

$$\frac{\partial \psi}{\partial \tau}+\frac{1}{2}\left(3 \frac{n_0}{V^4}-\frac{3 \bar{n}_0}{Q^2 V^4}-1\right) \psi \frac{\partial \psi}{\partial \xi}+\frac{1}{2} \frac{\partial^3 \psi}{\partial \xi^3}=0, \tag{1}$$

where $\psi$ is the wave potential, $\tau$ the normalized time and the other quantities are the same as defined in [7]. Now when $\bar{n}_0$, the density of negative ions, satisfies the equation

$$\frac{3 n_0}{V^4}-\frac{3 \bar{n}_0}{Q^2 V^4}-1=0, \tag{2}$$

the nonlinear term disappears from the l.h.s. of (1). The density $\bar{n}_0$, given by (2), is called the *critical density* of the negative ions. To derive the MKdV equation at the critical density of the negative ions in an inhomogeneous plasma we proceed as follows:

The equation of continuity and the equations of motion for the ions are

$$\frac{\partial n_i}{\partial t}+\frac{\partial}{\partial x}(n_i u_i)=0, \tag{3}$$

$$\frac{\partial u_i}{\partial t}+u_i \frac{\partial u_i}{\partial x}=g_i \frac{\partial \varphi}{\partial x}, \tag{4}$$

where $i$ takes the values $\alpha$ and $\beta$ for positive and negative ions, respectively, and $g_\alpha=-1$, $g_\beta=1/Q$, $Q$ being the ratio of negative ion mass $m_2$ to the positive ion mass $m_1$. Let $n_e$ denote the electron density, then the Boltzman and Poisson equation are given by

$$n_e=e^{\varphi}, \tag{5}$$

$$\frac{\partial^2 \varphi}{\partial x^2}=n_e+n_\beta-n_\alpha. \tag{6}$$

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To solve (3) and (4) we expand the quantities $n_i$, $n_e$, $u_i$ and $\varphi$ as follows:

$$\begin{aligned} n_i &= n_{i0} + \varepsilon n_{i1} + \varepsilon^2 n_{i2} + \dots, \\ u_i &= u_{i0} + \varepsilon u_{i1} + \varepsilon^2 u_{i2} + \dots, \\ n_e &= n_{e0} + \varepsilon n_{e1} + \varepsilon^2 n_{e2} + \dots, \\ \varphi &= \varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \dots, \end{aligned} \tag{7}$$

It is to be noted that we have considered a lowest order ion fluid velocity. This is because a spatial inhomogeneity in the lowest order implies a lowest order ion fluid velocity. To solve the above equations we use the reductive perturbation method [5]. The following stretched coordinates are suitable for a spatially inhomogeneous plasma:

$$\xi = \varepsilon \left[ \int^x \frac{\mathrm{d}x'}{\lambda_0(x')} - t \right], \quad \eta = \varepsilon^3 x, \tag{8}$$

where $\lambda_0(x)$ is the velocity of the moving frame to be determined later so that the theory becomes self consistent. The above coordinates are taken in order to derive the MKdV equation at the critical density of the negative ions.

Since $\lambda_0$ and $n_{i0}$ are functions of $x$ only, we have

$$\frac{\partial \lambda_0}{\partial \xi} = 0, \quad \frac{\partial n_{i0}}{\partial \xi} = 0. \tag{9}$$

Also from the second order equation we obtain

$$u_{i1} = -\frac{g_i \varphi_1}{P_i n_{i0}}, \tag{10}$$

$$n_{i1} = -g_i \frac{\varphi_1}{P_i^2 n_{i0}}, \tag{11}$$

where $P_i = \dfrac{\lambda_0 - u_{i0}}{n_{i0}}$. In deriving (10) and (11) we have used the boundary condition that the plasma is homogeneous at the boundary, that is for $|\xi| \to \infty$: $n_{i1} = u_{i1} = -g_i \varphi_1$, $n_{i0} = \lambda_0 = 1$ and $u_{i0} = \varphi_0 = 0$. This explains the absence of additional $\eta$ dependent functions in (10) and (11). Using Poisson's relation together with Boltzmann's equation we have

$$n_{e0} - \frac{1}{Q P_\beta^2 n_{\beta 0}} - \frac{1}{P_\alpha^2 n_{\alpha 0}} = 0, \tag{12}$$

where

$$n_{e0} = e^{\varphi_0}. \tag{13}$$

Equating terms of the next higher order we have

$$n_{\alpha 2} = \frac{1}{P_\alpha^2 n_{\alpha 0}} \varphi_2 + \frac{3}{2 P_\alpha^4 n_{\alpha 0}^3} \varphi_1^2 + \frac{\lambda_0}{P_\alpha n_{\alpha 0}} \xi \cdot \left[ \frac{u_{\alpha 0}}{P_\alpha} \frac{\partial u_{\alpha 0}}{\partial \eta} + \frac{1}{P_\alpha} \frac{\partial \varphi_0}{\partial \eta} + \frac{\partial}{\partial \eta} (n_{\alpha 0} u_{\alpha 0}) \right], \tag{14}$$

$$u_{\alpha 2} = \frac{\varphi_2}{P_\alpha n_{\alpha 0}} + \frac{1}{2 P_\alpha^3 n_{\alpha 0}^3} \varphi_1^2 + \frac{\lambda_0 \xi}{P_\alpha n_{\alpha 0}} \cdot \left[ u_{\alpha 0} \frac{\partial u_{\alpha 0}}{\partial \eta} + \frac{\partial \varphi_0}{\partial \eta} \right], \tag{15}$$

$$n_{\beta 2} = -\frac{\varphi_2}{P_\beta^2 n_{\beta 0} Q} + \frac{3 \varphi_1^2}{2 P_\beta^4 n_{\beta 0}^3 Q^2} + \frac{\lambda_0 \xi}{P_\beta n_{\beta 0}} \cdot \left[ \frac{u_{\beta 0}}{P_\beta} \frac{\partial u_{\beta 0}}{\partial \eta} - \frac{1}{Q P_\beta} \frac{\partial \varphi_0}{\partial \eta} + \frac{\partial}{\partial \eta} (n_{\beta 0} u_{\beta 0}) \right], \tag{16}$$

and

$$u_{\beta 2} = -\frac{\varphi_2}{P_\beta n_{\beta 0} Q} + \frac{1}{2 P_\beta^3 n_{\beta 0}^3 Q^2} \varphi_1^2 + \frac{\lambda_0 \xi}{P_\beta n_{\beta 0}} \cdot \left[ u_{\beta 0} \frac{\partial u_{\beta 0}}{\partial \eta} - \frac{1}{Q} \frac{\partial \varphi_0}{\partial \eta} \right]. \tag{17}$$

In deriving (14)–(17) we have once again used the condition of homogeneity at $|\xi| \to \infty$ together with the boundary conditions such that in the homogeneous case (14)–(17) become identical with (19) and (20) of [6], provided we take $\lambda_0 = V$.

Using Poisson's relation together with Boltzmann's equation (i.e., from (5) and (6)) we have

$$n_{e0} \left( \varphi_2 + \frac{\varphi_1^2}{2} \right) + n_{\beta 2} - n_{\alpha 2} = 0. \tag{18}$$

Putting the values of $n_{\alpha 2}$ and $n_{\beta 2}$ as obtained from (14) and (16) we have

$$\begin{aligned} &\left[ n_{e0} - \frac{1}{P_\beta^2 n_{\beta 0} Q} - \frac{1}{P_\alpha^2 n_{\alpha 0}} \right] \varphi_2 \\ &+ \left[ \frac{n_{e0}}{2} + \frac{3}{2 P_\beta^4 n_{\beta 0}^3 Q^2} - \frac{3}{2 P_\alpha^4 n_{\alpha 0}^3} \right] \varphi_1^2 + \frac{\lambda_0 \xi}{P_\beta n_{\beta 0}} \cdot \left[ \frac{u_{\beta 0}}{P_\beta} \frac{\partial u_{\beta 0}}{\partial \eta} - \frac{1}{Q} \frac{1}{P_\beta} \frac{\partial \varphi_0}{\partial \eta} + \frac{\partial}{\partial \eta} (u_{\beta 0} n_{\beta 0}) \right] \\ &- \frac{\lambda_0 \xi}{P_\alpha n_{\alpha 0}} \left[ \frac{u_{\alpha 0}}{P_\alpha} \frac{\partial u_{\alpha 0}}{\partial \eta} + \frac{1}{P_\alpha} \frac{\partial \varphi_0}{\partial \eta} + \frac{\partial}{\partial \eta} (u_{\alpha 0} n_{\alpha 0}) \right] = 0. \end{aligned} \tag{19}$$

Using (12) in (19) we have

$$\varphi_1^2 = \frac{2\lambda_0 \xi F}{n_{e0} + \dfrac{3}{P_\beta^4 n_{\beta 0}^3 Q^2} - \dfrac{3}{P_\alpha^4 n_{\alpha 0}^3}}, \tag{20}$$

where

$$F = \left[ \frac{u_{\alpha 0}}{P_\alpha^2 n_{\alpha 0}} \frac{\partial u_{\alpha 0}}{\partial \eta} - \frac{u_{\beta 0}}{P_\beta^2 n_{\beta 0}} \frac{\partial u_{\beta 0}}{\partial \eta} + \left( \frac{1}{P_\alpha^2 n_{\alpha 0}} + \frac{1}{P_\beta^2 n_{\beta 0} Q} \right) \frac{\partial \varphi_0}{\partial \eta} + \frac{1}{P_\alpha n_{\alpha 0}} \frac{\partial}{\partial \eta} (n_{\alpha 0} u_{\alpha 0}) - \frac{1}{P_\beta n_{\beta 0}} \frac{\partial}{\partial \eta} (n_{\beta 0} u_{\beta 0}) \right]. \tag{21}$$

It is seen from (20) that in the homogeneous case $\varphi_1$ vanishes identically unless the denominator is zero. In the inhomogeneous case, to keep $\varphi_1$ finite but indeterminate, both $F$ and the denominator must vanish. Hence we get $F = 0$ and also

$$n_{e0} + \frac{3}{P_\beta^4 n_{\beta 0}^3} - \frac{3}{P_\alpha^4 n_{\alpha 0}^3} = 0. \tag{22}$$

It is easily seen that in the homogeneous case (22) coincides with the critical density equation (2). Hence in the inhomogeneous case (22) may be considered as the equation determining the critical density of negative ions. The next higher order equations now give

$$\frac{\partial n_{\alpha 3}}{\partial \xi} - \frac{1}{P_\alpha^2 n_{\alpha 0}} \frac{\partial \varphi_3}{\partial \xi} = \frac{1}{P_\alpha n_{\alpha 0}} \left[ \frac{\partial}{\partial \xi} (n_{\alpha 1} u_{\alpha 2}) + \frac{\partial}{\partial \xi} (n_{\alpha 2} u_{\alpha 1}) \right] + \frac{1}{P_\alpha^2 n_{\alpha 0}} \left[ u_{\alpha 1} \frac{\partial u_{\alpha 2}}{\partial \xi} + u_{\alpha 2} \frac{\partial u_{\alpha 1}}{\partial \xi} \right] + \frac{\lambda_0}{P_\alpha n_{\alpha 0}} \left[ \frac{u_{\alpha 0}}{P_\alpha} \frac{\partial u_{\alpha 1}}{\partial \xi} + \frac{u_{\alpha 1}}{P_\alpha} \frac{\partial u_{\alpha 0}}{\partial \eta} + \frac{1}{P_\alpha} \frac{\partial \varphi_1}{\partial \eta} + \frac{\partial}{\partial \eta} (n_{\alpha 0} u_{\alpha 1}) + \frac{\partial}{\partial \eta} (n_{\alpha 1} u_{\alpha 0}) \right] \tag{23}$$

and a similar equation for $n_{\beta 3}$. For the Poisson and Boltzmann equations we have

$$n_{e3} = n_{e0} (\varphi_3 + \varphi_1 \varphi_2 + \tfrac{1}{6} \varphi_1^3) \tag{24}$$

and

$$\frac{\partial^2 \varphi_1}{\lambda_0^2 \partial \xi^2} = n_{e3} + n_{\beta 3} - n_{\alpha 3}. \tag{25}$$

Putting (24) in (25) and differentiating once,

$$\frac{\partial^3 \varphi}{\lambda_0^2 \partial \xi^3} = n_{e0} \left( \frac{\partial \varphi_3}{\partial \xi} + \frac{\partial}{\partial \xi} (\varphi_1 \varphi_2) + \tfrac{1}{2} \varphi_1^2 \frac{\partial \varphi_1}{\partial \xi} \right) + \frac{\partial n_{\beta 3}}{\partial \xi} - \frac{\partial n_{\alpha 3}}{\partial \xi}. \tag{26}$$

Putting the values of $\dfrac{\partial n_{\alpha_3}}{\partial \xi}$ and $\dfrac{\partial n_{\beta 3}}{\partial \xi}$ from (23) and using the relations (12) and (22) it can be easily seen that all higher order terms exactly cancel and we finally arrive at the modified KdV equation with $\xi$ dependent terms

$$\frac{\partial \varphi_1}{\partial \eta} + Z_1 \varphi_1^2 \frac{\partial \varphi_1}{\partial \eta} + Z_2/2\lambda_0^3 \frac{\partial^3 \varphi_1}{\partial \xi^3} + Z_3 \xi \frac{\partial \varphi_1}{\partial \xi} + Z_4 \varphi_1 = 0, \tag{27}$$

where

$$Z_1 = \left[ -n_{e0} + 15 \left( \frac{1}{P_\beta^6 n_{\beta 0}^5 Q^3} + \frac{1}{P_\alpha^6 n_{\alpha 0}^5} \right) \right] Z_2/4\lambda_0, \tag{28}$$

$$Z_3 = -S Z_2/2\lambda_0, \tag{29}$$

$$Z_4 = -(S + R) Z_2/2\lambda_0, \tag{30}$$

$$Z_2 = \left( \frac{u_{\beta 0}}{P_\beta^3 Q n_{\beta 0}^2} + \frac{1}{P_\beta^2 Q n_{\beta 0}} + \frac{u_{\alpha 0}}{P_\alpha^3 n_{\alpha 0}^2} + \frac{1}{P_2^2 n_{\alpha 0}} \right)^{-1}. \tag{31}$$

$R$ and $S$ can be written as

$$R = A_\alpha + A_\beta, \quad S = B_\alpha + B_\beta,$$

where

$$A_i = h_i g_i \left[ \frac{\lambda_0 u_{i0}}{P_1^2 n_{i0}} \frac{\partial}{\partial \eta} \left( \frac{1}{P_i n_{i0}} \right) + \frac{\lambda_0}{P_i^3 n_{i0}^2} \frac{\partial u_{i0}}{\partial \eta} + \frac{\lambda_0}{P_i n_{i0}} \frac{\partial}{\partial \eta} \left( \frac{1}{P_i} \right) + \frac{\lambda_0}{P_i n_{i0}} \frac{\partial}{\partial \eta} \left( \frac{u_{i0}}{P_i^2 n_{i0}} \right) \right] \tag{32}$$

and

$$B_i = -h_i g_i^2 \frac{3\lambda_0}{P_i^4 n_{i0}^3} \frac{\partial \varphi_0}{\partial \eta} - 3 g_i h_i \frac{\lambda_0 u_{i0}}{P_i^4 n_{i0}^3} \frac{\partial u_{i0}}{\partial \eta} - \frac{h_i g_i \lambda_0}{P_i^3 n_{i0}^3} \frac{\partial}{\partial \eta} (n_{i0} u_{i0}), \tag{33}$$

where $h_\alpha = 1$, $h_\beta = -1$ and $g_i$ is same as in (2).

An equation similar to (27) but with lower order nonlinearity was studied in detail by Roychoudhury et al. [8]. Explicit solution of (18) can be obtained in the case of $Z_3 = Z_4$, whence the equation can be reduced to the one discussed in [8]. To solve (18) we use a transformation similar to the one first given by Hirota [9].

We put

$$\varphi_1 = \sqrt{\frac{6\beta}{Z_1}}\, P(\eta)\, v(\varrho, \tau), \tag{34}$$

where

$$\beta = (Z_2/2\lambda_0^3)^{1/3}, \quad \varrho = \frac{P\,\xi}{\beta}, \quad \tau = \int_{\eta_0}^{\eta} P^3(\eta')\,\mathrm{d}\eta', \tag{35}$$

$$P = P_0 \exp\left(-\int_{\eta_0}^{\eta} Z_3(\eta')\,\mathrm{d}\eta'\right).$$

Then (18) is reduced to

$$v_\tau + 6v^2 v_\tau + v_{\varrho\varrho\varrho} + \frac{R}{P^3} v = 0. \tag{36}$$

When $R = 0$, (27) reduces to the standard form of the MKdV equation, and its one soliton solution is given by

$$v(\varrho, \tau) = A \operatorname{sech} A(\varrho - A^2\tau - \varrho_0). \tag{37}$$

It can be seen from (25) and (28) that as the density increases the amplitude decreases. The width and velocity of the soliton also get modified as a direct consequence of the inhomogeneity. In presence of the term $R/P^3$, (36) can be considered a perturbation problem. The solution can then be found following the Zakharov Shabat inverse scattering technique, which has been discussed in detail by Kaup [10]. Due to the presence of the term $R/P^3$, the parameters $A$ and $\tau_0$ in (28) will no more be constant but evolve slowly with $\tau$, for example

$$\frac{\partial A}{\partial \tau} = A(\tau_0)\, e^{-\frac{2R\tau}{P^3}}, \quad \frac{\partial}{\partial \tau}(A\,\varrho_0) + \frac{2R}{P^3} A\varrho_0 = A^3(\tau). \tag{38}$$

Hence for small $R$, one soliton solution of (18) can be easily obtained. It is also to be noted that the critical density of the negative ions will depend on the density through the terms $P_\alpha$ and $P_\beta$ as can be seen from (11).

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